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INTEGRABILITY AND BOUNDEDNESS OF LOCAL SOLUTIONS TO DOUBLY DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract. Integral and supremum estimates are proven for local solutions of doubly degenerate quasilinear parabolic equations. This is done with the aid of a local energy inequality and estimates in L_q^{weak} spaces.

1. Introduction and results. In this paper we shall obtain $L_{q,loc}(\Omega_T)$ and $L_{\infty,loc}(\Omega_T)$ estimates for local solutions of a class of doubly degenerate quasilinear parabolic equations modeled after

$$\frac{\partial}{\partial t}u - \operatorname{div}\left(|u|^{m-1}\,|\nabla u|^{p-2}\,\nabla u\right) = f(x,t) + \operatorname{div}\mathbf{g}(x,t). \tag{1}$$

If $m = 1$ and $p = 2$, this is simply the heat equation. If $m = 1$ and $p \neq 2$, we have the evolution p-Laplacian; if $m > 1$ and $p = 2$, we have the porous medium equation.

For our general model, let $\Omega \subseteq \mathbb{R}^N$ be a domain, let $T > 0$, and let $\Omega_T = \Omega \times (0,T)$. Consider a general equation of the form

$$\frac{\partial}{\partial t}u - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u) \tag{2}$$

where for almost every $(x,t,u,\mathbf{v}) \in \Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^N$ the following structure conditions hold.

H1) $1 < p < \infty$, $m \geq 1$, $\delta < \mu \equiv (\frac{N+2}{N})p + (m-1)$, $\Phi(u) \geq 0$, $\phi_i(x,t) \geq 0$, $C_i \geq 0$, $C_6 > 0$.

H2) $a(x,t,u,\mathbf{v})\cdot\mathbf{v} \geq \Phi(|u|)|\mathbf{v}|^p - C_3|u|^\delta - \phi_o(x,t)$.

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H3) $|a(x,t,u,\mathbf{v})| \le C_1\{\Phi(|u|)|\mathbf{v}|^p\}^{1-\frac{1}{p}} + C_4|u|^{\delta(1-\frac{1}{p})} + \phi_1(x,t)$.

H4) $|b(x,t,u,\mathbf{v})| \le C_2\{\Phi(|u|)|\mathbf{v}|^p\}^{1-\frac{1}{\delta}} + C_5|u|^{\delta-1} + \phi_2(x,t)$.

H5) $\Phi(u) \ge C_6 u^{m-1}$ for all $u \ge C_7$.

H6) $\phi_o \in L_{\alpha,loc}(\Omega_T)$, $\phi_1, \phi_2 \in L_{\beta,loc}(\Omega_T)$ for some $\alpha > 1$ and $\beta > \frac{\mu}{\mu-1}$.

On the solution itself, we assume the following:

H7) For every $0 \le t_1 < t_2 \le T$ and for every $\Omega' \Subset \Omega$

$$\operatorname*{ess\,sup}_{t_1<t<t_2} \int_{\Omega'} |u(x,t)|^2\,dx + \int_{t_1}^{t_2}\int_{\Omega'} \Phi(|u|)\,|\nabla u|^p\,dx\,dt < \infty,$$

H8) $u \in L_{r,loc}(\Omega_T)$ for some $r > \frac{N}{p}[2-p-(m-1)]$.

Finally, we assume that one of the following is satisfied:

H9a) $\nabla u \in L_{p,loc}(\Omega_T)$ and $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$, or

H9b) There are sequences of functions u_n, a_n, and b_n with $u_n \rightharpoonup u$ weakly in $L_{1,loc}(\Omega_T)$ so that $\frac{\partial}{\partial t}u_n - \operatorname{div} a_n(x,t,u_n,\nabla u_n) = b_n(x,t,u_n,\nabla u_n)$, so that H1–H8 are satisfied uniformly in n, and so that $\nabla u_n \in L_{p,loc}(\Omega_T)$ and $\phi_{1,n} \in L_{\frac{p}{p-1},loc}(\Omega_T)$ but not necessarily uniformly in n.

By a weak solution of (2), we mean a function u satisfying H7 so that

$$\iint_{\Omega_T} \{-u\psi_t + a(x,t,u,\nabla u)\cdot\nabla\psi\}\,dx\,dt = \iint_{\Omega_T} b(x,t,u,\nabla u)\psi\,dx\,dt \quad (3)$$

for all $\psi \in C_o^\infty(\Omega_T)$. Our main result is the following.

Theorem 1. *Let u be a weak solution of* (2)*, and suppose that* H1–H8 *and one of* H9 *are satisfied.*

If $\min\{\alpha,\beta\} > (N+p)/p$*, then* $u \in L_{\infty,loc}(\Omega_T)$;

if $\min\{\alpha,\beta\} = (N+p)/p$*, then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < \infty$;

if $\min\{\alpha,\beta\} < (N+p)/p$*, then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < q^*$*, where* $q^* = \min\big\{\frac{\mu}{1-(1-\frac{1}{\alpha})(1+\frac{p}{N})}, \frac{\mu-(1+\frac{p}{N})}{1-(1-\frac{1}{\beta})(1+\frac{p}{N})}\big\}$.

We remark that in the case that H9a is imposed, the resulting bounds are independent of $\|\nabla u\|_{L_{p,loc}(\Omega_T)}$ and $\|\phi_1\|_{L_{\frac{p}{p-1},loc}(\Omega_T)}$.

Doubly nonlinear equations of this type have been studied extensively, with the survey of Kalashnikov [14] providing a good starting point. The

problem of existence has been studied variously in [2, 3, 5, 6, 11, 12, 13, 22]. Questions on the Hölder continuity of solutions have been examined by Vespri and Porzio as well as Ivanov in [7, 8, 20, 23], and a Harnack-type estimate has been obtained by Vespri in [24]. Results on the boundedness of solutions have been obtained by Porzio in [19], and in Lieberman [17]; other results have been obtained in [9, 10, 16, 25, 26].

Our results improve upon what is known in a number of ways. First we are able to estimate the degree of local integrability of solutions which are not necessarily bounded; this is new and extends the results of [18] where such estimates were proven for singular and degenerate quasilinear equations with constant Φ. In [19], Porzio proved the boundedness of solutions to a similar class of problems under the restriction $\Phi(u) = u^{m-1}$, although the proof is only presented for the case where $p \geq [2-(m-1)]\frac{N}{N+2}$, i.e., where $\mu \geq 2$. Additionally, Lieberman in [17] studied doubly degenerate diffusions of a very general form, but only provided quantitative estimates for solutions already known to be bounded.

A few remarks about our result and its hypotheses are now in order. First note that the choice of the number $\mu = \left(\frac{N+2}{N}\right)p + (m-1)$ is motivated by the following extensions of the usual Sobolev embedding theorem.

Proposition 2. *Let $\Omega_T = \Omega \times (0,T) \subseteq \mathbb{R}^n \times \mathbb{R}$ be a domain, let $p, m \geq 1$, and let u be a function with* $\operatorname{supp} u(\cdot,t) \Subset \Omega$ *for almost every t satisfying*

$$\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2\,dx + \iint_{\Omega_T} |u|^{m-1}|\nabla u|^p\,dx\,dt < \infty. \tag{4}$$

Then there is a constant C depending only on N, p, and m so that

$$\iint_{\Omega_T} |u(x,t)|^{(\frac{N+2}{N})p+(m-1)}\,dx\,dt \leq C\Big(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2\,dx\Big)^{\frac{p}{N}} \Big(\iint_{\Omega_T} |u|^{m-1}|\nabla u|^p\,dx\,dt\Big). \tag{5}$$

Proposition 3. *Let $\Omega_T = \Omega \times (0,T) \subseteq \mathbb{R}^n \times \mathbb{R}$ be a domain and let u be a function satisfying* H7. *Then $u \in L_{\mu,loc}(\Omega_T)$.*

Proofs of these results are presented in Section 6. Next, we point out that the assumption H8 is necessary to prove boundedness, even in the case where $\Phi = 1$; this was noted in [4]. Note also that if $p > \frac{2-(m-1)}{N+2}$ then we can simply set $r = 2$.

Finally we would like to discuss the assumption H9. Our proof proceeds by using a smoothed cutoff version of $(u-k)_+$ as a test function to obtain an energy estimate. This estimate is then iterated to obtain our results. The assumption H9 is made to ensure that we can, in fact, use a smoothed cutoff version of $(u-k)_+$ as a test function. Consequently, these estimates are used only qualitatively and this is why when H9a is imposed the bounds on the solution are independent of both $\|\nabla u\|_{p,loc}$ and $\|\phi_o\|_{\frac{p}{p-1},loc}$.

The condition $\phi \in L_{\frac{p}{p-1},loc}(\Omega_T)$ is somewhat natural in view of the fact that a similar condition was imposed by Ladyzhenskaya, Solonnikov, and Ural'ceva in [15, Chapter 5, Theorem 2.1, Equation 2.2] in their study of nondegenerate quasilinear equations.

However, in light of the structure of our equation, the assumption $\nabla u \in L_{p,loc}(\Omega_T)$ does not appear natural, and its appearance could be questioned. Moreover, because $m \geq 1$, we know that $(u-k)_+ \in L_{p,loc}(0,T;W^1_{p,loc}(\Omega))$ for all $k > C_7$, so that it appears that this requirement is unnecessary to use a smoothed cutoff version of $(u-k)_+$ as a test function. In fact, if u were sufficiently regular in time, say if $u_t \in L_{2,loc}(\Omega_T)$, then we could simply use a cutoff version of $(u-k)_+$ as our test function. In the linear, quasilinear, and degenerate quasilinear cases, this additional regularity in time is unnecessary, and is avoided by using cutoff versions of $(u_h-k)_+$ as test functions in the Steklov averaged equations, and then passing to the limit as $h \downarrow 0$, where $u_h(x,t) = \frac{1}{h}\int_t^{t+h} u(x,s)\,ds$ is the Steklov average. In the doubly degenerate case, however, this approach is problematic. In particular, though H5 and H7 are sufficient to guarantee that $\nabla(u-k)_+ \in L_{p,loc}(\Omega_T)$ for $k \geq C_7$, they do not appear to be sufficient to guarantee that $\nabla(u_h-k)_+ \in L_{p,loc}(\Omega_T)$, nor do they appear to be sufficient to ensure convergence as $h \downarrow 0$. The difficulty is the lack of any control on the behavior of u in time that would allow us to relate $(u-k)_+$ and $(u_h-k)_+$.

It is possible, however, to replace H9 with some additional regularity in time, but without going to the extreme of requiring that $u_t \in L_{2,loc}(\Omega_T)$. Instead of H9, we could impose

H9a′) $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$ and for all $\Omega' \times (t_1,t_2) \Subset \Omega_T$

$$\lim_{h\downarrow 0} \int_{t_1}^{t_2} \int_{\Omega'} \frac{|u(x,t+h)-u(x,t)|^2}{h}\,dx\,dt = 0 \tag{6}$$

or

H9b′) There are sequences of functions u_n, a_n, and b_n with $u_n \rightharpoonup u$ weakly in $L_{1,loc}(\Omega_T)$ so that

$$\frac{\partial}{\partial t}u_n - \operatorname{div} a_n(x,t,u_n,\nabla u_n) = b_n(x,t,u_n,\nabla u_n),$$

so that H1–H8 are satisfied uniformly in n, and so that (6) obtains for each function u_n, and $\phi_{1,n} \in L_{\frac{p}{p-1},loc}(\Omega_T)$, but not necessarily uniformly in n

Then we have the following.

Corollary 1.1. *Let u be a solution of* (2) *and suppose that* H1–H8 *and one of* H9′ *are satisfied. Then the conclusions of Theorem* 1 *remain valid.*

The condition H9′ is not as unnatural as it might first seem. In particular, in the book of Ladyzhenskaya, Solonnikov, and Ural'ceva [15], they show that weak solutions of the linear [15, Chapter 3, Lemma 4.1] or quasilinear [15, Chapter 5, Theorem 6.7] Dirichlet problems satisfy (6) under reasonable assumptions on the structure of the equation and on the data.

In our case, the assumption H9′ lets us use a Steklov averaged and cutoff version of $(u-k)_+$ as a test function; this avoids the difficulty of proving that $\nabla(u_h - k)_+ \in L_{p,loc}(\Omega_T)$, while the regularity imposed by H9′ lets us pass to the limit in the u_t term.

2. Sketch of Proof. As we have already noted, the main tool in our investigation is a local energy estimate. In particular, we have the following.

Proposition 4. *Let u be a weak solution of* (2) *and suppose that* H1–H7 *and* H9a *hold. Then for any $Q_R(x_o,t_o) \equiv B_R(x_o)\times(t_o - R^p, t_o) \Subset \Omega_T$, for any $0<\sigma<1$, and for any k with $k \geq C_7$, we have*

$$\begin{aligned}
&\Big\{\iint_{Q_{\sigma R}} (u \mp k)_\pm^\mu \, dx\, dt\Big\}^{\frac{N}{N+p}} \leq \frac{C}{(1-\sigma)^p R^p}\iint_{Q_R}\{(u\mp k)_\pm^2 + (u\mp k)_\pm^p\}\,dx\,dt \\
&\quad + \frac{C}{(1-\sigma)^p R^p}\iint_{Q_R} |u|^{p+m-1}\chi[(u\mp k)_\pm > 0]\,dx\,dt \qquad (7)\\
&\quad + C\iint_{Q_R}|u|^\delta \chi[(u\mp k)_\pm>0]\,dx\,dt + C\Big\{\frac{\|\phi_1\|_{L_\beta(Q_R)}}{(1-\sigma)R} + \|\phi_2\|_{L_\beta(Q_R)}\Big\} \\
&\quad \times \Big(\iint_{Q_R}(u\mp k)_\pm^{\frac{\beta}{\beta-1}}\,dx\,dt\Big)^{1-\frac{1}{\beta}} + C\|\phi_o\|_{L_\alpha(Q_R)}\{\operatorname{meas}[(u\mp k)_\pm>0]\}^{1-\frac{1}{\alpha}}
\end{aligned}$$

for some constant C depending only on C_i, N, p, and m.

Corollary 4.1. *The results of Proposition* 4 *remain true if* H9a′ *replaces* H9a.

These results are proven by using smoothed cutoff versions of $(u \mp k)_\pm$ as test functions; the details of the proof are provided in Section 5.

Once we have these energy estimates, the proof follows the same lines as it did in the quasilinear case [18]. Indeed, if we know that $u \in L_{q,loc}(\Omega_T)$ for some q, we can use the energy inequality to obtain an estimate of the form

$$\operatorname{meas}\{(x,t) \in Q_{\sigma R} : |u(x,t)| > k\} \le C(\|u\|_{L_q(Q_R)})(\frac{1}{k})^{\nu(q)} \tag{8}$$

for some function $\nu(q)$, where C is independent of k. This then shows us that $|u|_{L^{\text{weak}}_{\nu(q)}(Q_{\sigma R})} < \infty$. Recall the definitions of the spaces $L^{\text{weak}}_q(\mathcal{U})$: a measurable function is an element of $L^{\text{weak}}_q(\mathcal{U})$ if and only if

$$|f|_{L^{\text{weak}}_q(\mathcal{U})} \equiv \sup_{k>0} k\,[\operatorname{meas}\{x \in \mathcal{U} : |f(x)| > k\}]^{\frac{1}{q}} < \infty. \tag{9}$$

Although $|u|_{L^{\text{weak}}_q(\mathcal{U})}$ is not a norm, it is a quasinorm. Further, if $\mathcal{U}$ is bounded, then $L_q(\mathcal{U}) \subset L^{\text{weak}}_q(\mathcal{U}) \subset L_{q'}(\mathcal{U})$ for all $q' < q$. Indeed,

$$|f|^q_{L^{\text{weak}}_q(\mathcal{U})} = \sup_{k>0} k^q \operatorname{meas}[|f| > k] \le \sup_{k>0} \int_{\mathcal{U}} |f|^q \chi[|f| > k]\,dx \le \int_{\mathcal{U}} |f|^q\,dx$$

while

$$\|f\|^{q'}_{L'_q(\mathcal{U})} = q' \int_0^\infty k^{q'-1} \operatorname{meas}[|f| > k] dk \le q' \operatorname{meas} \mathcal{U} + q' |f|_{L^{\text{weak}}_q(\mathcal{U})} \int_1^\infty k^{q'-q-1} dk.$$

For further details, see [1] or [21, IX.4]. The energy inequality in the form (8) then tells us that if $u \in L_{q_o,loc}(\Omega_T)$, then $u \in L^{\text{weak}}_{\nu(q_o),loc}(\Omega_T)$ for some $\nu(q)$. Consequently, $u \in L_{q,loc}(\Omega_T)$ for all $q < (\nu \circ \nu \circ \cdots \circ \nu)(q_o)$, which upon iteration yields our $L_{q,loc}(\Omega_T)$ estimates for $q < \infty$.

To prove the boundedness of solutions, we shall use the local integrability guaranteed by these $L_{q,loc}(\Omega_T)$ and the usual DeGiorgi iteration techniques, coupled with an interpolation in the case $\mu \le 2$.

In the cases where H9b or H9b′ is imposed, we proceed as described above for each approximating function u_n. Because H1–H8 are satisfied uniformly in n, we obtain $L_{q,loc}(\Omega_T)$ or $L_{\infty,loc}(\Omega_T)$ estimates for u_n that are independent of n. The weak convergence $u_n \rightharpoonup u$ is then used to identify the limit.

3. The $L_{q,loc}(\Omega_T)$ estimates. Suppose that $u \in L_{q,loc}(\Omega_T)$ for some $q \geq \mu$. We begin by estimating the left side of (7) with the $(u-k)_+$ choice as

$$\Big(\iint_{Q_{\sigma R}} (u-k)_+^{\mu} \, dx \, dt \Big)^{\frac{N}{N+p}} \geq \Big(k^{\mu} \operatorname*{meas}_{Q_{\sigma R}} [u > 2k] \Big)^{\frac{N}{N+p}}. \tag{10}$$

As a consequence

$$\begin{aligned} \big(k^{\mu} \operatorname*{meas}_{Q_{\sigma R}} [u > 2k] \big)^{\frac{N}{N+p}} &\leq C \iint_{Q_R} \big\{ (u-k)_+^2 + (u-k)_+^p \big\} \, dx \, dt \\ &\quad + C \iint_{Q_R} \big(|u|^{p+m-1} + |u|^{\delta} \big) \chi[u > k] \, dx \, dt \\ &\quad + C \Big(\iint_{Q_R} (u-k)_+^{\frac{\beta}{\beta-1}} dx \, dt \Big)^{1-\frac{1}{\beta}} + C \big(\operatorname*{meas}_{Q_R} [u > k] \big)^{1-\frac{1}{\alpha}} \end{aligned} \tag{11}$$

for a constant C independent of k. Next, note that if $\theta < q$, then

$$\begin{aligned} \iint_{Q_R} (u-k)_+^{\theta} \, dx \, dt &\leq \iint_{Q_R} |u|^{\theta} \chi[u > k] \, dx \, dt \\ &\leq \Big(\iint_{Q_R} |u|^q \, dx \, dt \Big)^{\frac{\theta}{q}} \Big(\operatorname*{meas}_{Q_R} [u > k] \Big)^{1-\frac{\theta}{q}} \\ &\leq \|u\|_{L_q(Q_R)}^{\theta} \Big(\frac{1}{k^q} |u|_{L_q^{\mathrm{weak}}(Q_R)}^q \Big)^{1-\frac{\theta}{q}} \leq \|u\|_{L_q(Q_R)}^q \Big(\frac{1}{k} \Big)^{q-\theta}. \end{aligned} \tag{12}$$

Note also that

$$\big(\operatorname*{meas}_{Q_R} [u > k] \big)^{1-\frac{1}{\alpha}} \leq \big(\frac{1}{k^q} |u|_{L_q^{\mathrm{weak}}(Q_R)}^q \big)^{1-\frac{1}{\alpha}} \leq \|u\|_{L_q(Q_R)}^{q\left(1-\frac{1}{\alpha}\right)} \big(\frac{1}{k} \big)^{q\left(1-\frac{1}{\alpha}\right)}.$$

Together, these imply that

$$\begin{aligned} \big(k^{\mu} \operatorname*{meas}_{Q_{\sigma R}} [u > 2k] \big)^{\frac{N}{N+p}} &\leq C \|u\|_{L_q(Q_R)}^q \Big\{ \big(\frac{1}{k} \big)^{q-2} + \big(\frac{1}{k} \big)^{q-p} + \big(\frac{1}{k} \big)^{q-(p+m-1)} \\ &\quad + \big(\frac{1}{k} \big)^{q-\delta} \Big\} + C \|u\|_{L_q(Q_R)}^{q\left(1-\frac{1}{\beta}\right)} \big(\frac{1}{k} \big)^{q\left(1-\frac{1}{\beta}\right)-1} + C \|u\|_{L_q(Q_R)}^{q\left(1-\frac{1}{\alpha}\right)} \big(\frac{1}{k} \big)^{q\left(1-\frac{1}{\alpha}\right)}. \end{aligned} \tag{13}$$

If we repeat this process with the $(u+k)_-$ choice in Proposition 4 or Corollary 4.1, we obtain the estimate

$$\begin{aligned} \operatorname*{meas}_{Q_{\sigma R}} [|u| > 2k] \leq C \Big\{ &\big(\frac{1}{k} \big)^{(q-2)\left(1+\frac{p}{N}\right)+\mu} + \big(\frac{1}{k} \big)^{(q-\delta)\left(1+\frac{p}{N}\right)+\mu} \\ &+ \big(\frac{1}{k} \big)^{[q\left(1-\frac{1}{\beta}\right)-1]\left(1+\frac{p}{N}\right)+\mu} + \big(\frac{1}{k} \big)^{q\left(1-\frac{1}{\alpha}\right)\left(1+\frac{p}{N}\right)+\mu} \Big\} \end{aligned} \tag{14}$$

for all $k \geq C_7$, where C remains independent of k; note that we have used the fact that

$$\big(\frac{1}{k}\big)^{q-p} + \big(\frac{1}{k}\big)^{q-(p+m-1)} \leq \big(C_7^{\delta-p} + C_7^{\delta-(p+m-1)}\big)\big(\frac{1}{k}\big)^{q-\delta} \tag{15}$$

for all $k \geq C_7$. Set

$$\nu_1(q) = (q-2)(1+\frac{p}{N}) + \mu, \quad \nu_3(q) = [q(1-\frac{1}{\beta}) - 1](1+\frac{p}{N}) + \mu,$$
$$\nu_2(q) = (q-\delta)(1+\frac{p}{N}) + \mu, \quad \nu_4(q) = q(1-\frac{1}{\alpha})(1+\frac{p}{N}\big) + \mu,$$

and $\nu(q) = \min\{\nu_1(q), \nu_2(q), \nu_3(q), \nu_4(q)\}$. We shall study the iterations $q_o, \nu(q_o), \nu(\nu(q_o)), \dots$ where $q_o = \max\{2, \mu, r\}$ is chosen so that our hypotheses guarantee that $u \in L_{q_o,loc}(\Omega_T)$. This will then imply that $u \in L_{q,loc}(\Omega_T)$ for any $q < (\nu \circ \nu \circ \cdots \circ \nu)(q_o)$.

Case 1: ν_1. Note that $\nu_1(q) > q$ if and only if $q > \frac{N}{p}[2(1+\frac{p}{N}) - \mu] = \frac{N}{p}[2-p-(m-1)]$. As a consequence, the sequence $q_o, \nu_1(q_o), \nu_1(\nu_1(q_o)), \dots$ will tend to infinity if $q_o > \frac{N}{p}[2-p-(m-1)]$. Indeed, the above shows that the sequence is monotone increasing; if it tended to a finite limit, that limit would have to be a fixed point of ν_1 greater than q_o. Since no such fixed point exists, we conclude that the sequence tends to infinity. That $q_o > \frac{N}{p}[2-p-(m-1)]$, is an immediate consequence of H8 and the fact that $q_o \geq r$.

Case 2: ν_2. We see that $\nu_2(q) > q$ if and only if $q > \frac{N}{p}\big[\delta\big(1+\frac{p}{N}\big) - \mu\big]$. Then since H1 implies $\delta < \mu$, we see that $\frac{N}{p}[\delta(1+\frac{p}{N}) - \mu] < \frac{N}{p}[\mu(1+\frac{p}{N}) - \mu] = \mu \leq q_o$, and we conclude that the sequence $q_o, \nu_2(q_o), \nu_2(\nu_2(q_o)), \dots$ tends to infinity.

Case 3: ν_3. Here the situation is somewhat different. We can write $\nu_3(q)$ as $\nu_3(q) = q(1+\frac{p}{N})(1-\frac{1}{\beta}) + [\mu - (1+\frac{p}{N})]$. Since $\mu > (1+\frac{p}{N})$, we see that the sequence $q_o, \nu_3(q_o), \nu_3(\nu_3(q_o)), \dots$ tends to infinity whenever $(1+\frac{p}{N})(1-\frac{1}{\beta}) \geq 1$, or equivalently, if $\beta \geq \frac{N+p}{p}$. Conversely suppose $\beta < \frac{N+p}{p}$; then $\nu_3(q) > q$ if and only if $q < \frac{\mu-(1+\frac{p}{N})}{1-(1-\frac{1}{\beta})(1+\frac{p}{N})}$ so that the sequence $q_o, \nu_3(q_o), \nu_3(\nu_3(q_o)), \dots$ tends to $q_\beta^* = \frac{\mu-(1+\frac{p}{N})}{1-(1-\frac{1}{\beta})(1+\frac{p}{N})}$.

Case 4: ν_4. Here $\nu_4(q) = q(1-\frac{1}{\alpha})(1+\frac{p}{N}) + \mu$. As in case 3, if $(1-\frac{1}{\alpha})(1+\frac{p}{N}) \geq 1$, or equivalently if $\alpha \geq \frac{N+p}{p}$, then the sequence $q_o, \nu_4(q_o), \nu_4(\nu_4(q_o)), \dots$ tends to infinity. On the other hand, if $\alpha < \frac{N+p}{p}$, then the sequence $q_o, \nu_4(q_o), \nu_4(\nu_4(q_o)), \dots$ tends to $q_\alpha^* = \frac{\mu}{1-(1-\frac{1}{\alpha})(1+\frac{p}{N})}$.

4. The $L_{\infty,loc}(\Omega_T)$ estimates. In this section we shall show the boundedness of solutions when $\alpha, \beta > \frac{N+p}{p}$. Let $Q_\rho(x_o, t_o) \Subset \Omega_T$, let $0 < \sigma < 1$, and for each integer n set $\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho$, and $k_n = k(1 - \frac{1}{2^{n+1}})$ where $k \geq C_7$ shall be chosen later. Define $Q^n = Q_{\rho_n}$ so that $Q^o = Q_\rho$ and $Q^\infty = Q_{\sigma\rho}$, and apply Proposition 4 with k replaced by k_{n+1}, where $Q_R = Q^n$, and where $Q_{\sigma R} = Q^{n+1}$. Doing so, we obtain

$$\begin{aligned}
&\Big\{ \iint_{Q^{n+1}} (u - k_{n+1})_+^\mu \, dx\, dt \Big\}^{\frac{N}{N+p}} \\
&\leq \frac{C\, 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} \Big\{ (u - k_{n+1})_+^2 + (u - k_{n+1})_+^p \Big\} \, dx\, dt \\
&+ \frac{C\, 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} |u|^{p+m-1} \chi[u > k_{n+1}] \, dx\, dt \\
&+ C \iint_{Q^n} |u|^\delta \chi[u > k_{n+1}] \, dx\, dt \\
&+ \frac{C\, 2^n}{(1-\sigma)\rho} \Big(\iint_{Q^n} (u - k_{n+1})_+^{\frac{\beta}{\beta-1}} dx\, dt \Big)^{1-\frac{1}{\beta}} + C \Big(\operatorname{meas} A_{n+1} \Big)^{1-\frac{1}{\alpha}}
\end{aligned} \tag{16}$$

for a constant C independent of n and k, where $A_{n+1} = \big\{(x,t) \in Q^n : u(x,t) > k_{n+1}\big\}$. Before we estimate the right side of this inequality, we note the following useful fact. Given $\theta > 1$, we have

$$\begin{aligned}
\iint_{Q^n} (u - k_n)_+^\theta \, dx\, dt &\geq \iint_{Q^n} (u - k_n)_+^\theta \chi[u > k_{n+1}] \, dx\, dt \\
&\geq (k_{n+1} - k_n)^\theta \operatorname{meas} A_{n+1} \geq \big(\frac{k}{2^{n+2}}\big)^\theta \operatorname{meas} A_{n+1}
\end{aligned} \tag{17}$$

so that

$$\operatorname{meas} A_{n+1} \leq \big(\frac{2^{n+2}}{k}\big)^\theta \iint_{Q^n} (u - k_n)_+^\theta \, dx\, dt.$$

Our proof now proceeds in two separate cases, depending on the value of the parameter μ.

Case 1: $\mu > 2$. We shall use (16) to obtain an iterative inequality for

$$Y_n \equiv \fint\!\!\!\fint_{Q^n} (u - k_n)_+^\mu \, dx\, dt = \frac{1}{\operatorname{meas} Q^n} \iint_{Q^n} (u - k_n)_+^\mu \, dx\, dt. \tag{18}$$

To estimate the first term on the right side of (16), note that Hölder's inequality and the fact that $\mu > 2$ give us the estimate

$$\begin{aligned}
&\frac{C\,2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^2\,dx\,dt \qquad (19)\\
&\qquad\le \frac{C\,2^{np}}{(1-\sigma)^p\rho^p}\Big(\iint_{Q^n}(u-k_{n+1})_+^\mu\,dx\,dt\Big)^{\frac{2}{\mu}}\Big(\operatorname{meas}A_{n+1}\Big)^{1-\frac{2}{\mu}}\\
&\le \frac{C\,2^{np}}{(1-\sigma)^p\rho^p}\Big(\frac{2^{n+2}}{k}\Big)^{\mu-2}\iint_{Q^n}(u-k_n)_+^\mu\,dx\,dt\\
&\le \frac{C}{(1-\sigma)^p}\rho^N 2^{n(p+\mu-2)}\Big(\frac{1}{k}\Big)^{\mu-2}Y_n.
\end{aligned}$$

We then use similar arguments to obtain the estimate

$$\frac{C\,2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}(u-k_{n+1})_+^p\,dx\,dt \le \frac{C}{(1-\sigma)^p}\rho^N 2^{n\mu}\Big(\frac{1}{k}\Big)^{\mu-p}Y_n. \qquad (20)$$

As for the third and fourth terms on the right side of (16), notice that if $(x,t)\in[u>k_{n+1}]$, then $\frac{u}{u-k_n}\le\frac{k_{n+1}}{k_{n+1}-k_n}$. As a consequence, for each θ we have $|u|^\theta\chi[u>k_{n+1}]\le 2^{(n+2)\theta}(u-k_n)_+^\theta\chi[u>k_{n+1}]$. Thus,

$$\begin{aligned}
&\frac{C\,2^{np}}{(1-\sigma)^p\rho^p}\iint_{Q^n}|u|^{p+m-1}\chi[u>k_{n+1}]\,dx\,dt\\
&\quad\le \frac{C\,2^{np}}{(1-\sigma)^p\rho^p}2^{(n+2)(p+m-1)}\iint_{Q^n}(u-k_n)_+^{p+m-1}\chi[u>k_{n+1}]\,dx\,dt\\
&\qquad\le \frac{C}{(1-\sigma)^p}\rho^N 2^{n(\mu+p)}\Big(\frac{1}{k}\Big)^{\mu-p-(m-1)}Y_n \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
\iint_{Q^n}|u|^\delta\chi[u>k_{n+1}]\,dx\,dt &\le 2^{(n+2)\delta}\iint_{Q^n}(u-k_n)_+^\delta\chi[u>k_{n+1}]\,dx\,dt\\
&\le C\rho^{N+p}2^{n\mu}\Big(\frac{1}{k}\Big)^{\mu-\delta}Y_n. \qquad (22)
\end{aligned}$$

Because H6 requires that $\beta>\frac{\mu}{\mu-1}$, we know that $\mu>\frac{\beta}{\beta-1}$ and we can use Hölder's inequality to estimate the fourth term on the right side of (16) as

$$\begin{aligned}\frac{C\,2^n}{(1-\sigma)\rho}&\Big(\iint_{Q^n}(u-k_{n+1})_+^{\frac{\beta}{\beta-1}}dx\,dt\Big)^{1-\frac{1}{\beta}}\\ &\le \frac{C\,2^n}{(1-\sigma)\rho}\Big(\iint_{Q^n}(u-k_{n+1})_+^{\mu}dx\,dt\Big)^{\frac{1}{\mu}}\Big(\operatorname{meas}A_{n+1}\Big)^{1-\frac{1}{\beta}-\frac{1}{\mu}} \qquad (23)\\ &\le \frac{C}{(1-\sigma)}\rho^{(N+p)(1-\frac{1}{\beta})-1}2^{n\mu(1-\frac{1}{\beta})}\Big(\frac{1}{k}\Big)^{\mu(1-\frac{1}{\beta})-1}Y_n^{1-\frac{1}{\beta}}.\end{aligned}$$

Finally, we can estimate the last term on the right side of (16):

$$\begin{aligned}C\Big[\operatorname{meas}A_{n+1}\Big]^{1-\frac{1}{\alpha}}&\le C\Big\{\Big(\frac{2^{n+2}}{k}\Big)^{\mu}\iint_{Q^n}(u-k_n)_+^{\mu}\,dx\,dt\Big\}^{1-\frac{1}{\alpha}}\\ &\le C\rho^{(N+p)(1-\frac{1}{\alpha})}2^{n\mu(1-\frac{1}{\alpha})}\Big(\frac{1}{k}\Big)^{\mu(1-\frac{1}{\alpha})}Y_n^{1-\frac{1}{\alpha}}.\end{aligned} \qquad (24)$$

If we combine the estimates from (19), (20), (21), (22), and (23), we find that

$$\begin{aligned}\Big(\rho^{N+p}Y_{n+1}\Big)^{\frac{N}{N+p}}&\le \frac{C\rho^N2^{n(p+\mu-2)}}{(1-\sigma)^pk^{\mu-2}}Y_n\\ &+\frac{C\rho^N2^{n\mu}}{(1-\sigma)^pk^{\mu-p}}Y_n+\frac{C\rho^N2^{n(\mu+p)}}{(1-\sigma)^pk^{\mu-p-(m-1)}}Y_n+\frac{C\rho^{N+p}2^{n\mu}}{k^{\mu-\delta}}Y_n\\ &+\frac{C\rho^{(N+p)(1-\frac{1}{\beta})-1}2^{n\mu(1-\frac{1}{\beta})}}{(1-\sigma)k^{\mu(1-\frac{1}{\beta})-1}}Y_n^{1-\frac{1}{\beta}}+\frac{C\rho^{(N+p)(1-\frac{1}{\alpha})}2^{n\mu(1-\frac{1}{\alpha})}}{k^{\mu(1-\frac{1}{\alpha})}}Y_n^{1-\frac{1}{\alpha}}.\end{aligned}$$

Note that hypothesis H6 implies that $\mu(1-\frac{1}{\beta})-1>0$, so that if we fix ρ and σ, we obtain the inequality $Y_{n+1}\le CB^n[Y_n^{1+\frac{p}{N}}+Y_n^{(1-\frac{1}{\alpha})(1+\frac{p}{N})}+Y_n^{(1-\frac{1}{\beta})(1+\frac{p}{N})}]$ where B and C are independent of n and k. The assumptions that $\alpha,\beta>\frac{N+p}{p}$ then imply $(1-\frac{1}{\alpha})(1+\frac{p}{N})>1$ and $(1-\frac{1}{\alpha})(1+\frac{p}{N})>1$ so that the usual rules on fast geometric convergence imply that $Y_n\to 0$ if Y_o is sufficiently small, dependent on B, C, p, N, α, and β. However

$$Y_o=\fint\!\!\!\!\!\!\int_{Q_\rho}\Big(u-\frac{k}{2}\Big)_+^{\mu}\,dx\,dt \qquad (25)$$

so that we may choose k so large as to guarantee that $Y_n\to 0$; in particular there exists some k so that

$$Y_\infty=\fint\!\!\!\!\!\!\int_{Q_{\sigma\rho}}(u-k)_+^{\mu}\,dx\,dt=0 \qquad (26)$$

making u locally bounded above. Similar considerations for $(u+k)_-$ show that u is bounded below, and our result follows.

Case 2: $\mu \le 2$. We can not proceed as we did before in this case because the estimate (19) will no longer be valid. Let $\lambda > \max\{2, \mu\}$ be chosen later, and set

$$Y_n = \fint\!\!\!\fint_{Q^n} (u-k_n)_+^\lambda \, dx\, dt. \tag{27}$$

This is finite thanks to our $L_{q,loc}(\Omega_T)$ estimates. Now given any $\Lambda > \lambda > \mu$, the interpolation inequality implies

$$\begin{aligned} Y_{n+1} = \fint\!\!\!\fint_{Q^{n+1}} (u-k_{n+1})_+^\lambda \, dx\, dt &\le \frac{1}{\operatorname{meas} Q^{n+1}} \Big(\iint_{Q^{n+1}} (u-k_{n+1})_+^\Lambda dx\, dt \Big)^{\frac{\lambda}{\Lambda}\theta} \\ &\quad \times \Big(\iint_{Q^{n+1}} (u-k_{n+1})_+^\mu \, dx\, dt \Big)^{\frac{\lambda}{\mu}(1-\theta)} \end{aligned}$$

where $\theta = \frac{\frac{1}{\mu}-\frac{1}{\lambda}}{\frac{1}{\mu}-\frac{1}{\Lambda}} = \frac{\Lambda}{\lambda}\frac{\lambda-\mu}{\Lambda-\mu}$. Thus,

$$\iint_{Q^{n+1}} (u-k_{n+1})_+^\mu \, dx\, dt \ge C\rho^{(N+p)\frac{\Lambda-\mu}{\Lambda-\lambda}} \Big(\frac{1}{\|u\|_{L_\Lambda(Q_\rho)}} \Big)^{\Lambda\frac{\lambda-\mu}{\Lambda-\lambda}} Y_{n+1}^{\frac{\Lambda-\mu}{\Lambda-\lambda}} \tag{28}$$

which estimates the left side of (16). The right side of that inequality is estimated in a fashion similar to case 1; in particular letting C depend on σ, we have

$$\begin{aligned} &\frac{C\,2^{np}}{(1-\sigma)^p\rho^p} \iint_{Q^n} (u-k_{n+1})_+^2 \, dx\, dt \le C\rho^N 2^{n(p+\lambda-2)} \Big(\frac{1}{k}\Big)^{\lambda-2} Y_n, \\ &\frac{C\,2^{np}}{(1-\sigma)^p\rho^p} \iint_{Q^n} (u-k_{n+1})_+^p \, dx\, dt \le C\rho^N 2^{n\lambda} \Big(\frac{1}{k}\Big)^{\lambda-p} Y_n, \\ &\frac{C\,2^{np}}{(1-\sigma)^p\rho^p} \iint_{Q^n} |u|^{p+m-1} \chi[u>k_{n+1}] \, dx\, dt \le C\rho^N 2^{n(\lambda+p)} \Big(\frac{1}{k}\Big)^{\lambda-p-(m-1)} Y_n, \\ &\iint_{Q^n} |u|^\delta \, dx\, dt \le C\rho^{N+p} 2^{n\lambda} \Big(\frac{1}{k}\Big)^{\lambda-\delta} Y_n, \end{aligned}$$

$$\begin{aligned} &\frac{C\,2^n}{(1-\sigma)\rho} \Big(\iint_{Q^n} (u-k_{n+1})_+^{\frac{\beta}{\beta-1}} dx\, dt \Big)^{1-\frac{1}{\beta}} \\ &\le C\rho^{(N+p)(1-\frac{1}{\beta})-1} 2^{n\lambda(1-\frac{1}{\beta})} \Big(\frac{1}{k}\Big)^{\lambda(1-\frac{1}{\beta})-1} Y_n^{1-\frac{1}{\beta}}, \\ &C[\operatorname{meas} A_{n+1}]^{1-\frac{1}{\alpha}} \le C\rho^{(N+p)(1-\frac{1}{\alpha})} 2^{n\lambda(1-\frac{1}{\alpha})} Y_n^{1-\frac{1}{\alpha}}. \end{aligned}$$

Combining these gives us the estimate

$$\begin{aligned}
&\Big\{\rho^{(N+p)\left(\frac{\Lambda-\mu}{\Lambda-\lambda}\right)}\Big(\frac{1}{\|u\|_{L_\Lambda(Q_\rho)}}\Big)^{\Lambda\frac{\lambda-\mu}{\Lambda-\lambda}}Y_{n+1}^{\frac{\Lambda-\mu}{\Lambda-\lambda}}\Big\}^{\frac{N}{N+p}}\\
&\le \frac{C\rho^N 2^{n(p+\lambda-2)}}{k^{\lambda-2}}Y_n+\frac{C\rho^N 2^{n\lambda}}{k^{\lambda-p}}Y_n+\frac{C\rho^N 2^{n(\lambda+p)}}{k^{\lambda-p-(m-1)}}Y_n+\frac{C\rho^{N+p}2^{n\lambda}}{k^{\lambda-\delta}}Y_n\\
&+\frac{C\rho^{(N+p)(1-\frac{1}{\beta})-1}2^{n\lambda(1-\frac{1}{\beta})}}{k^{\lambda(1-\frac{1}{\beta})-1}}Y_n^{1-\frac{1}{\beta}}+\frac{C\rho^{(N+p)(1-\frac{1}{\alpha})}2^{n\lambda(1-\frac{1}{\alpha})}}{k^{\lambda(1-\frac{1}{\alpha})}}Y_n^{1-\frac{1}{\alpha}}.
\end{aligned}$$

Because $\lambda>\mu$, we know that $\lambda\big(1-\frac{1}{\beta}\big)-1>0$; thus there are constants B and C independent of k so that

$$Y_{n+1}\le CB^n\Big[Y_n^{(1+\frac{p}{N})(\frac{\Lambda-\lambda}{\Lambda-\mu})}+Y_n^{(1-\frac{1}{\alpha})(1+\frac{p}{N})(\frac{\Lambda-\lambda}{\Lambda-\mu})}+Y_n^{(1-\frac{1}{\beta})(1+\frac{p}{N})(\frac{\Lambda-\lambda}{\Lambda-\mu})}\Big]. \quad (29)$$

Because $\lim_{\Lambda\to\infty}\frac{\Lambda-\lambda}{\Lambda-\mu}=1$ for all values of λ and μ, we can choose $\Lambda>\lambda$ so that $(1-\frac{1}{\alpha})(1+\frac{p}{N})(\frac{\Lambda-\lambda}{\Lambda-\mu})>1$, and $(1-\frac{1}{\beta})(1+\frac{p}{N})(\frac{\Lambda-\lambda}{\Lambda-\mu})>1$. Thus with λ and Λ now fixed, we can proceed as we did in case one, and our claim follows.

5. Proof of the energy inequality. In this section we shall provide proofs of Proposition 4 and Corollary 4.1.

Proof of Proposition 4. Let $(x_o,t_o)\in\Omega_T$; modulo a translation we may assume that (x_o,t_o) is the origin. Set $Q_R=B_R(0)\times(-R^p,0)$, and suppose that $Q_R\Subset\Omega_T$. Choose τ so that $-R^p<\tau<0$, and set $Q_R^\tau=B_R(0)\times(-R^p,\tau)$. Fix $0<\sigma<1$, and let $\zeta\in C^\infty(Q_R)$ be a cutoff function so that $0\le\zeta\le1$, with $\zeta(x,t)=1$ when $(x,t)\in Q_{\sigma R}$, so that $\zeta(x,t)=0$ near $|x|=R$ or $t=-R^p$, and so that $|\nabla\zeta|^p+|\zeta_t|\le 4/(1-\sigma)^pR^p$. Let $\zeta_j\in C_0^\infty(Q_R)$ be chosen so that $\zeta_j\to\zeta$ uniformly in Q_R^τ, so that $\zeta_j=0$ if $t>\tau-\frac{1}{j}$ and so that $\zeta_j=\zeta$ if $t<\tau-\frac{2}{j}$. Let J_η be a family of symmetric mollifying kernels in space-time for each $\eta>0$, and let $f_\eta=J_\eta*f$ denote the space-time mollification. Fix $\eta>0$, $k\ge C_7$, and define $\psi(x,t)=\big\{(u_\eta-k)_+\zeta_j^p\big\}_\eta$. If η is sufficiently small, then $\psi\in C_0^\infty(\Omega_T)$; thus we can use it as a test function in (3) to obtain

$$\begin{aligned}
\iint_{\Omega_T}&-u\frac{\partial}{\partial t}\Big\{(u_\eta-k)_+\zeta_j^p\Big\}_\eta\,dx\,dt\\
&+\iint_{\Omega_T}a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u_\eta-k)_+\zeta_j^p\Big\}_\eta\,dx\,dt
\end{aligned}$$

$$= \iint_{\Omega_T} b(x,t,u,\nabla u)\Big\{(u_\eta - k)_+\zeta_j^p\Big\}_\eta \, dx\, dt;$$

i.e., $I_1 + I_2 = I_3$. We shall investigate each term in turn.

Since the mollifier is symmetric, we can rewrite I_1 and integrate by parts to obtain

$$I_1 = -\iint_{\Omega_T} u_\eta \frac{\partial}{\partial t}\Big\{(u_\eta - k)_+\zeta_j^p\Big\} dx\, dt = \frac{1}{2}\iint_{Q_R^\tau} \Big\{\frac{\partial}{\partial t}(u_\eta - k)_+^2\Big\}\zeta_j^p \, dx\, dt.$$

Send $j \to \infty$ and then integrate by parts once more to obtain

$$\lim_{j\to\infty} I_1 = \frac{1}{2}\int_{B_R} (u_\eta - k)_+^2 \zeta^p \, dx\Big|_\tau - \frac{p}{2}\iint_{Q_R^\tau} (u_\eta - k)_+^2 \zeta^{p-1}\zeta_t \, dx\, dt.$$

The assumption H7 gives us sufficient integrability to pass to the limit in η; doing so we obtain for almost every τ

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_1 = \frac{1}{2}\int_{B_R} (u - k)_+^2 \zeta^p \, dx\Big|_\tau - \frac{p}{2}\iint_{Q_R^\tau} (u - k)_+^2 \zeta^{p-1}\zeta_t \, dx\, dt. \quad (30)$$

Next we turn to I_2, which can be written as

$$I_2 = \iint_{\Omega_T} a_\eta^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u_\eta - k)_+\zeta_j^p\Big\} \, dx\, dt.$$

The boundedness and smoothness of the mollifications lets us send $j \to \infty$; next we want to pass to the limit as $\eta \downarrow 0$. First note that for any $\Omega' \times (t_1, t_2) \Subset \Omega_T$ the structure condition H3 implies

$$\int_{t_1}^{t_2}\int_{\Omega'} |a(x,t,u,\nabla u)|^{\frac{p}{p-1}} dx\, dt \le C\int_{t_1}^{t_2}\int_{\Omega'} \Phi(|u|)|\nabla u|^p \, dx\, dt$$
$$+ C\int_{t_1}^{t_2}\int_{\Omega'} |u|^\delta \, dx\, dt + C\int_{t_1}^{t_2}\int_{\Omega'} |\phi_1(x,t)|^{\frac{p}{p-1}} dx\, dt < \infty \quad (31)$$

thanks to H7, H9a, and Proposition 3. Together with the usual properties of mollifiers this implies that $a_\eta^i(x,t,u,\nabla u) \xrightarrow{\eta\downarrow 0} a^i(x,t,u,\nabla u)$ in $L_{\frac{p}{p-1}}(Q_R^\tau)$ for each i. On the other hand, we can use H7, H9a, and Proposition 3 to see that $\frac{\partial}{\partial x_i}\big\{(u_\eta - k)_+\zeta^p\big\} \xrightarrow{\eta\downarrow 0} \frac{\partial}{\partial x_i}\big\{(u - k)_+\zeta^p\big\}$ in $L_p(Q_R^\tau)$. Together these let us pass to the limit as $\eta \downarrow 0$ in I_2 and conclude that

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty}\iint_{Q_R^\tau} a_\eta^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u_\eta - k)_+\zeta^p\Big\} \, dx\, dt \quad (32)$$
$$= \iint_{Q_R^\tau} a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u - k)_+\zeta^p\Big\} \, dx\, dt.$$

We remark that this is the only stage where the assumption H9a is used. Expand (32) and use the structure conditions H2 and H3 to obtain

$$\begin{aligned}\lim_{\eta\downarrow 0}\lim_{j\to 0} I_2 \geq & \iint_{Q_R^\tau} \Phi(|u|)\,|\nabla(u-k)_+|^p\zeta^p\,dx\,dt \\ & - C_3\iint_{Q_R^\tau} |u|^\delta\chi[u>k]\zeta^p\,dx\,dt - \iint_{Q_R^\tau}\phi_o\chi[u>k]\,dx\,dt \\ & - C_1 p\iint_{Q_R^\tau}\left\{\Phi(|u|)|\nabla(u-k)_+|^p\zeta^p\right\}^{1-\frac{1}{p}}(u-k)_+|\nabla\zeta|\,dx\,dt \\ & - C_4 p\iint_{Q_R^\tau}\left\{|u|^\delta\chi[u>k]\zeta^p\right\}^{1-\frac{1}{p}}(u-k)_+|\nabla\zeta|\,dx\,dt \\ & - p\iint_{Q_R^\tau}\phi_1(u-k)_+\zeta^{p-1}|\nabla\zeta|\,dx\,dt.\end{aligned}$$

Applying Young's inequality, we discover

$$\begin{aligned}\lim_{\eta\downarrow 0}\lim_{j\to 0} I_2 \geq & \frac{3}{4}\iint_{Q_R^\tau}\Phi(|u|)\,|\nabla(u-k)_+|^p\zeta^p\,dx\,dt \\ & - C\iint_{Q_R^\tau}|u|^\delta\chi[u>k]\zeta^p\,dx\,dt - C\iint_{Q_R^\tau}(u-k)_+^p|\nabla\zeta|^p\,dx\,dt \\ & - \iint_{Q_R^\tau}\phi_o\chi[u>k]\,dx\,dt - C\iint_{Q_R^\tau}\phi_1(u-k)_+\zeta^{p-1}|\nabla\zeta|\,dx\,dt\end{aligned} \tag{33}$$

for some constant C depending only on C_i and p.

Lastly we turn to I_3, which we can write as

$$I_3 = \iint_{\Omega_T} b_\eta(x,t,u,\nabla u)(u_\eta - k)_+\zeta_j^p\,dx\,dt.$$

The boundedness of the mollifications lets us send $j\to\infty$; further, if $\Omega'\times(t_1,t_2)\Subset\Omega_T$, the structure condition H4 implies

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega'}|b(x,t,u,\nabla u)|^{\frac{\mu}{\mu-1}}dx\,dt \leq & C\int_{t_1}^{t_2}\int_{\Omega'}\left\{\Phi(|u|)|\nabla u|^p\right\}^{\frac{\mu}{\mu-1}\frac{\delta-1}{\delta}}dx\,dt \\ & + C\int_{t_1}^{t_2}\int_{\Omega'}|u|^{\mu\frac{\delta-1}{\mu-1}}\,dx\,dt + C\int_{t_1}^{t_2}\int_{\Omega'}|\phi_2|^{\frac{\mu}{\mu-1}}\,dx\,dt<\infty\end{aligned} \tag{34}$$

because the facts that $(\frac{\mu}{\mu-1})(\frac{\delta-1}{\delta}) < 1$ and $\frac{\delta-1}{\mu-1} < 1$ let us use H6, H7, and Proposition 3. As a consequence, we know that $b_\eta(x,t,u,\nabla u) \xrightarrow{\eta\downarrow 0}$

$b(x,t,u,\nabla u)$ in $L_{\frac{\mu}{\mu-1}}(Q_R^\tau)$. On the other hand, Proposition 3 implies that $u \in L_{\mu,loc}(\Omega_T)$ so that $(u_\eta - k)_+\zeta^p \xrightarrow{\eta\downarrow 0} (u-k)_+\zeta^p$ in $L_\mu(Q_R^\tau)$. Together these allow us to pass to the limit as $\eta \downarrow 0$ in I_3 to obtain

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty}\iint_{Q_R^\tau} b_\eta(x,t,u,\nabla u)(u_\eta-k)_+\zeta_j^p\,dx\,dt = \iint_{Q_R^\tau} b(x,t,u,\nabla u)(u-k)_+\zeta^p\,dx\,dt. \quad (35)$$

Structure condition H4 then tells us that

$$\lim_{\eta\downarrow 0}\lim_{j\to\infty} I_3 \le C_2\iint_{Q_R^\tau}\left\{\Phi(|u|)|\nabla(u-k)_+|^p\zeta^p\right\}^{1-\frac{1}{\delta}}(u-k)_+\zeta^{\frac{p}{\delta}}dx\,dt + C_5\iint_{Q_R^\tau}|u|^{\delta-1}(u-k)_+\zeta^p\,dx\,dt + \iint_{Q_R^\tau}\phi_2(u-k)_+\zeta^p\,dx\,dt. \quad (36)$$

Young's inequality, (30), (33) and (36) then imply

$$\begin{aligned}\frac{1}{2}\int_{B_R}(u-k)_+^2\zeta^p\,dx\Big|_\tau &+ \frac{1}{2}\iint_{Q_R^\tau}\Phi(|u|)|\nabla(u-k)_+|^p\zeta^p\,dx\,dt \quad (37)\\ &\le C\iint_{Q_R^\tau}(u-k)_+^2\zeta^{p-1}\zeta_t\,dx\,dt + C\iint_{Q_R^\tau}(u-k)_+^p|\nabla\zeta|^p\,dx\,dt\\ &\quad + C\iint_{Q_R^\tau}|u|^\delta\chi[u>k]\zeta^p\,dx\,dt + C\iint_{Q_R^\tau}\phi_o\chi[u>k]\zeta^p\,dx\,dt\\ &\quad + C\iint_{Q_R^\tau}\phi_1(u-k)_+\zeta^{p-1}|\nabla\zeta|\,dx\,dt + C\iint_{Q_R^\tau}\phi_2(u-k)_+\zeta^p\,dx\,dt\end{aligned}$$

for almost every τ, where C depends only on C_i and p. Set $\kappa = \max\{1, \frac{2}{p}\} \ge 1$; then using H5 and the choice $k \ge C_7$, we obtain

$$\begin{aligned}\iint_{Q_R^\tau}\Phi(|u|)|\nabla(u-k)_+|^p\zeta^p\,dx\,dt &\ge C_6\iint_{Q_R^\tau}|u|^{m-1}|\nabla(u-k)_+|^p\zeta^p\,dx\,dt\\ &\ge C\iint_{Q_R^\tau}\left|(u-k)_+\zeta^\kappa\right|^{m-1}\left|\nabla\left\{(u-k)_+\zeta^\kappa\right\}\right|^p dx\,dt\\ &\quad - C\iint_{Q_R^\tau}|u|^{p+m-1}\chi[u>k]|\nabla\zeta|^p\zeta^{(\kappa-1)p}\,dx\,dt. \quad (38)\end{aligned}$$

Substitute this into (37), and use the fact that $-R^p < \tau < 0$ is arbitrary, to obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-R^p<\tau<0} \int_{B_R} \left|(u-k)_+\zeta^\kappa\right|^2 dx\Big|_\tau \\
&\qquad\qquad + \iint_{Q_R^\tau} \left|(u-k)_+\zeta^\kappa\right|^{m-1} \left|\nabla\left\{(u-k)_+\zeta^\kappa\right\}\right|^p dx\,dt \\
&\le \frac{C}{(1-\sigma)^p R^p} \iint_{Q_R^\tau} \left\{(u-k)_+^2 + (u-k)_+^p + |u|^{p+m-1}\chi[u>k]\right\} dx\,dt \\
&\qquad + C\iint_{Q_R^\tau} |u|^\delta \chi[u>k]\zeta^p\,dx\,dt \\
&\qquad + C\left\{\frac{\|\phi_1\|_{L_\beta(Q_R)}}{(1-\sigma)R} + \|\phi_2\|_{L_\beta(Q_R)}\right\}\left(\iint_{Q_R^\tau} (u-k)_+^{\frac{\beta}{\beta-1}}\,dx\,dt\right)^{1-\frac{1}{\beta}} \\
&\qquad + C\|\phi_o\|_{L_\alpha(Q_R)}\left(\operatorname*{meas}_{Q_R}[u>k]\right)^{1-\frac{1}{\alpha}}.
\end{aligned}$$

The result then follows after an application of Proposition 2 to the function $(u-k)_+\zeta^\kappa$, at least in the $(u-k)_+$ case. The result in the $(u+k)_-$ case is proven in the same fashion.

Proof of Corollary 4.1. To prove Corollary 4.1 we shall use the test function $\tilde{\psi}(x,t) \equiv \left\{(u-k)_+\zeta_j^p\right\}_{\bar{h}}$, where $k \ge C_7$, where $h > 0$ is a positive parameter, and where we use the Steklov averages

$$f_h(x,t) = \frac{1}{h}\int_t^{t+h} f(x,s)\,ds \quad \text{and} \quad f_{\bar{h}}(x,t) = \frac{1}{h}\int_{t-h}^t f(x,s)\,ds. \tag{39}$$

Examining (3), we see that density arguments and our hypotheses imply that we can use $\tilde{\psi}$ as a test function if $\operatorname{supp}\tilde{\psi} \Subset \Omega_T$ and $\tilde{\psi} \in W_2^1(0,T;L_2(\Omega)) \cap L_p(0,T;W_p^1(\Omega))$. As this is satisfied, we then find that

$$\begin{aligned}
&\iint_{\Omega_T} -u\frac{\partial}{\partial t}\left\{(u-k)_+\zeta_j^p\right\}_{\bar{h}}\,dx\,dt \\
&\qquad + \iint_{\Omega_T} a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\left\{(u-k)_+\zeta_j^p\right\}_{\bar{h}}\,dx\,dt \\
&\qquad\qquad = \iint_{\Omega_T} b(x,t,u,\nabla u)\left\{(u-k)_+\zeta_j^p\right\}_{\bar{h}}\,dx\,dt;
\end{aligned} \tag{40}$$

i.e., $\tilde{I}_1 + \tilde{I}_2 = \tilde{I}_3$. We estimate $\tilde{I}_2$ and $\tilde{I}_3$ in the same fashion as I_2 and I_3. Indeed, if we use the facts that $a^i(x,t,u,\nabla u) \in L_{\frac{p}{p-1},loc}(\Omega_T)$ and $(u-k)_+ \in L_{p,loc}(0,T;W^1_{p,loc}(\Omega))$, we can use the boundedness and convergence of ζ_j, together with the usual properties of the Steklov average, to conclude that

$$\lim_{h\downarrow 0}\lim_{j\to\infty}\tilde{I}_2 = \iint_{\Omega_T} a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u-k)_+\zeta^p\Big\}\,dx\,dt, \tag{41}$$

while the fact that $b(x,t,u,\nabla u) \in L_{\frac{\delta}{\delta-1},loc}(\Omega_T)$ and Proposition 3 imply that

$$\lim_{h\downarrow 0}\lim_{j\to\infty}\tilde{I}_3 = \iint_{\Omega_T} b(x,t,u,\nabla u)(u-k)_+\zeta^p\,dx\,dt. \tag{42}$$

We now turn our attention to $\tilde{I}_1$; for notational simplicity we shall write $u^+(t) \equiv (u-k)_+(\cdot,t)$ and $u^-(t) \equiv (u-k)_-(\cdot,t)$. Because

$$\frac{\partial\tilde{\psi}}{\partial t}(x,t) = \frac{1}{h}\big\{(u-k)_+(x,t)\zeta_j^p(x,t) - (u-k)_+(x,t-h)\zeta_j^p(x,t-h)\big\},$$

we have

$$\tilde{I}_1 = -\frac{1}{h}\iint_{\Omega_T} u(t)\big\{u^+(t)\zeta_j^p(t) - u^+(t-h)\zeta_j^p(t-h)\big\}dx\,dt$$

so that if we send $j\to\infty$, we find

$$\lim_{j\to\infty}\tilde{I}_1 = -\frac{1}{h}\iint_{\Omega_T} u(t)u^+(t)\zeta^p(t)\,dx\,dt + \frac{1}{h}\iint_{\Omega_T} u(t)u^+(t-h)\zeta^p(t-h)\,dx\,dt. \tag{43}$$

Make a change of variables in the second integral and add the results to obtain

$$\begin{aligned}\lim_{j\to\infty}\tilde{I}_1 &= \frac{1}{h}\iint_{\Omega_T}[u(t+h)-u(t)]u^+(t)\zeta^p(t)\,dx\,dt\\ &= \frac{1}{h}\iint_{\Omega_T}[(u-k)(\cdot,t+h)-(u-k)(\cdot,t)]u^+(t)\zeta^p(t)\,dx\,dt.\end{aligned} \tag{44}$$

Thus

$$\begin{aligned}\lim_{j\to\infty}\tilde{I}_1 = \frac{1}{h}\iint_{\Omega_T}&[u^+(t+h)-u^+(t)]u^+(t)\zeta^p(t)\,dx\,dt\\ &-\frac{1}{h}\iint_{\Omega_T}[u^-(t+h)-u^-(t)]u^+(t)\zeta^p(t)\,dx\,dt\end{aligned} \tag{45}$$

so that

$$\lim_{j\to\infty}\tilde{I}_1 = \frac{1}{2h}\iint_{\Omega_T}\{[u^+(t+h)]^2-[u^+(t)]^2\}\zeta^p(t)\,dx\,dt + \frac{1}{2h}\iint_{\Omega_T}|u^+(t+h)-u^+(t)|^2\zeta^p(t)\,dx\,dt - \frac{1}{h}\iint_{\Omega_T}[u^-(t+h)-u^-(t)]u^+(t)\zeta^p(t)\,dx\,dt = \tilde{I}_1^{(1)}+\tilde{I}_1^{(2)}+\tilde{I}_1^{(3)}.$$

We estimate each of these in turn.

Properties of Steklov averages let us write $\tilde{I}_1^{(1)}$ as

$$\tilde{I}_1^{(1)} = \frac{1}{2}\iint_{\Omega_T}\Big\{\frac{\partial}{\partial t}[(u-k)_+^2]_h\Big\}\zeta^p(t)\,dx\,dt; \tag{46}$$

integration by parts then implies

$$\tilde{I}_1^{(1)} = \frac{1}{2}\int_\Omega[(u-k)_+^2]_h\zeta^p\,dx\Big|_\tau - \frac{p}{2}\iint_{\Omega_T}[(u-k)_+^2]_h\zeta^{p-1}\zeta_t\,dx\,dt. \tag{47}$$

Using H7 and properties of the Steklov average, we can pass to the limit as $h\downarrow 0$ to obtain for almost every τ the relation

$$\lim_{h\downarrow 0}\tilde{I}_1^{(1)} = \frac{1}{2}\int_\Omega(u-k)_+^2\zeta^p\,dx\Big|_\tau - \frac{p}{2}\iint_{\Omega_T}(u-k)_+^2\zeta^{p-1}\zeta_t\,dx\,dt. \tag{48}$$

To estimate $\tilde{I}_1^{(2)}$, first note that

$$\begin{aligned}|u^+(t+h)-u^+(t)| &= |(u-k)_+(\cdot,t+h)-(u-k)_+(\cdot,t)|\\ &\le |(u-k)(\cdot,t+h)-(u-k)(\cdot,t)| \le |u(\cdot,t+h)-u(\cdot,t)|.\end{aligned} \tag{49}$$

Thus H9a′ implies

$$\lim_{h\downarrow 0}\Big|\tilde{I}_1^{(2)}\Big| \le \frac{1}{2}\lim_{h\downarrow 0}\iint_{Q_R^\tau}\frac{|u(x,t+h)-u(x,t)|^2}{h}\,dx\,dt = 0. \tag{50}$$

Finally, we turn to $\tilde{I}_1^{(3)}$; because $u^-(t)u^+(t)=0$, we have

$$\begin{aligned}\tilde{I}_1^{(3)} &= -\frac{1}{h}\iint_{\Omega_T}u^-(t+h)u^+(t)\zeta^p(t)\,dx\,dt\\ &= \frac{1}{h}\iint_{\Omega_T}\Big\{u^-(t+h)\chi[u(t)>k]\Big\}\Big\{-u^+(t)\chi[u(t+h)<k]\Big\}\zeta^p\,dx\,dt\\ &= \frac{1}{h}\iint_{\Omega_T}\Big\{[u^-(t+h)-u^-(t)]\chi[u(t)>k]\Big\}\end{aligned}$$

$$\times \Big\{[u^+(t+h) - u^+(t)]\chi[u(t+h) < k]\Big\}\zeta^p \, dx \, dt.$$

Consequently (49) coupled with a similar statement for u^- implies that

$$\lim_{h\downarrow 0} \left|\tilde{I}_1^{(3)}\right| \leq \lim_{h\downarrow 0} \iint_{Q_R^\tau} \frac{|u(x,t+h) - u(x,t)|^2}{h} \, dx \, dt = 0.$$

Combining these estimates for $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_3$, we find that

$$\begin{aligned} \frac{1}{2}\int_{B_R} (u-k)_+^2 \zeta^p \, dx\Big|_\tau - \frac{p}{2}\iint_{Q_R^\tau} (u-k)_+^2 \zeta^{p-1}\zeta_t \, dx \, dt \\ + \iint_{\Omega_T} a^i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\Big\{(u-k)_+\zeta^p\Big\} \, dx \, dt \\ = \iint_{\Omega_T} b(x,t,u,\nabla u)(u-k)_+\zeta^p \, dx \, dt, \end{aligned}$$

and the proof proceeds as did the proof of Proposition 4.

6. Proofs of Proposition 2 and Proposition 3.

Proof of Proposition 2. We begin this section by presenting the proof of Proposition 2. The proof proceeds differently depending on the relative values of p and N, so we break the proof up into cases.

Case 1: $p < N$. First we suppose that $p < N$. Hölder's inequality then implies that

$$\begin{aligned} \int_\Omega |u(x,t)|^{\frac{N+2}{N}p+(m-1)} \, dx &= \int_\Omega |u(x,t)|^{\frac{2p}{N}} |u(x,t)|^{(1+\frac{m-1}{p})p} \, dx \\ &\leq \Big(\int_\Omega |u(x,t)|^2 \, dx\Big)^{\frac{p}{N}} \Big(\int_\Omega \Big||u|^{1+\frac{m-1}{p}}\Big|^{\frac{Np}{N-p}} dx\Big)^{\frac{N-p}{N}}. \quad (51) \end{aligned}$$

Apply the usual Sobolev embedding inequality to find that, for each t,

$$\int_\Omega |u(x,t)|^{\frac{N+2}{N}p+(m-1)} \, dx \leq C\Big(\int_\Omega |u(x,t)|^2 \, dx\Big)^{\frac{p}{N}} \Big(\int_\Omega \Big|\nabla|u|^{1+\frac{m-1}{p}}\Big|^p dx\Big).$$

Our result follows after integrating both sides of this inequality in t.

Case 2: $p \geq N$. Now we handle the case where $p \geq N$. We begin by noting that

$$\int_\Omega |u(x,t)|^{\frac{N+2}{N}p+(m-1)} \, dx = \int_\Omega \Big||u(x,t)|^{1+\frac{m-1}{p}}\Big|^{\frac{\frac{N+2}{N}p+(m-1)}{1+\frac{m-1}{p}}} dx.$$

For notational simplicity, set

$$q = \frac{\frac{N+2}{N}p + (m-1)}{1 + \frac{m-1}{p}} > 1. \tag{52}$$

The full version of the Sobolev embedding Theorem [4, Chapter 1, Theorem 2.1] then implies that

$$\begin{aligned}\int_\Omega |u(x,t)|^\mu \, dx \, dt &= \int_\Omega \Big| |u(x,t)|^{1+\frac{m-1}{p}} \Big|^q \, dx \\ &\le C\Big(\int_\Omega |u(x,t)|^{(1+\frac{m-1}{p})s} \, dx \Big)^{\frac{q}{s}(1-\theta)} \Big(\int_\Omega \Big| \nabla |u(x,t)|^{1+\frac{m-1}{p}} \Big|^p \, dx \Big)^{\frac{q\theta}{p}}\end{aligned} \tag{53}$$

where

$$1 \le s \le q, \tag{54}$$

$$\theta = \big(\frac{1}{s} - \frac{1}{q}\big)\big(\frac{1}{N} + \frac{1}{s} - \frac{1}{p}\big)^{-1}, \tag{55}$$

$$0 \le \theta \le \frac{Np}{Np + s(p-N)}. \tag{56}$$

What happens next depends on the value of $1 + \frac{m-1}{p}$.

Case 2A: $p \ge N$ and $1 + \frac{m-1}{p} \le 2$. In this case, set $s = \frac{2}{1+\frac{m-1}{p}}$ and note that $1 \le s = \frac{2}{1+\frac{m-1}{p}} \le \frac{\frac{N+2}{N}p+(m-1)}{1+\frac{m-1}{p}} = q$. Further, using (55) to calculate θ, we find that $\theta = \frac{N(1+\frac{m-1}{p})}{2+N(1+\frac{m-1}{p})}$, and consequently, $0 \le \theta \le \frac{N(1+\frac{m-1}{p})}{2+N(1+\frac{m-1}{p})-2\frac{N}{p}} = \frac{Np}{Np+s(p-N)}$ as needed.

Substitute this choice of s into (53); then because a lengthy direct calculation shows us that

$$\frac{q}{s}(1-\theta) = \frac{p}{N} \quad \text{and} \quad \frac{q\theta}{p} = 1 \tag{57}$$

we can conclude

$$\int_\Omega |u(x,t)|^\mu \, dx \le C\Big(\int_\Omega |u(x,t)|^2 \, dx \Big)^{\frac{p}{N}} \Big(\int_\Omega \Big| \nabla |u|^{1+\frac{m-1}{p}} \Big|^p \, dx \Big) \tag{58}$$

so that our result will follow after integrating in t.

Case 2B: $p \geq N$ and $1 + \frac{m-1}{p} > 2$. In this case we set $s = 1$. The definition of θ then implies that

$$\theta = \frac{2\frac{p}{N} + (p-1)(1+\frac{m-1}{p})}{(\frac{p}{N} + p - 1)(\frac{2}{N} + (1+\frac{m-1}{p}))}, \tag{59}$$

and direct computation shows us that

$$\frac{Np}{Np + (p-N)} - \theta = \frac{1 + \frac{m-1}{p}}{(\frac{p}{N} + p - 1)(\frac{2}{N} + (1+\frac{m-1}{p}))} > 0 \tag{60}$$

so that conditions (54) and (56) are satisfied.

Apply (53) with these choices to conclude

$$\int_\Omega |u(x,t)|^\mu \, dx \leq C\Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}} \, dx\Big)^{q(1-\theta)} \times \Big(\int_\Omega \Big| \nabla |u(x,t)|^{1+\frac{m-1}{p}} \Big|^p \, dx\Big)^{\frac{q\theta}{p}}. \tag{61}$$

Because $\mu > 1 + \frac{m-1}{p} > 2$, we can apply the interpolation inequality to the first factor in (61), giving us

$$\Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}} \, dx\Big)^{\frac{1}{1+\frac{m-1}{p}}} \leq \Big(\int_\Omega |u(x,t)|^2 \, dx\Big)^{\frac{\omega}{2}} \Big(\int_\Omega |u(x,t)|^\mu \, dx\Big)^{\frac{1-\omega}{\mu}} \tag{62}$$

where $0 < \omega < 1$ is chosen to satisfy

$$\frac{1}{1+\frac{m-1}{p}} = \frac{\omega}{2} + \frac{1-\omega}{\mu}. \tag{63}$$

If we substitute (62) into (61), we find that

$$\int_\Omega |u(x,t)|^\mu \, dx \leq C\Big(\int_\Omega |u(x,t)|^2 \, dx\Big)^A \Big(\int_\Omega \Big| \nabla |u|^{1+\frac{m-1}{p}} \Big|^p \, dx\Big)^B \tag{64}$$

where another lengthy direct computation shows us that

$$A = \frac{\frac{\omega}{2}(1+\frac{m-1}{p})q(1-\theta)}{1 - \frac{1-\omega}{\mu}(1+\frac{m-1}{p})q(1-\theta)} = \frac{p}{N} \tag{65}$$

and

$$B = \frac{q\theta}{p}\frac{1}{1-\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q(1-\theta)} = 1 \tag{66}$$

so that Proposition 2 follows after an integration in time.

Next we need to prove Proposition 3. It shall follow from the following variant of Proposition 2:

Proposition 5. *Let $\Omega_T = \Omega \times (0,T) \subseteq \mathbb{R}^n \times \mathbb{R}$ be a bounded, piecewise-smooth domain, let $p, m \geq 1$, and let u be a function satisfying*

a) $\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2\,dx = \gamma < \infty$ *and*

b) $\iint_{\Omega_T} |u|^{m-1}|\nabla u|^p\,dx\,dt < \infty$.

Then $u \in L_\mu(\Omega_T)$.

Proof. We proceed in much the same fashion as we did in the proof of Proposition 2 by splitting the problem into cases.

Case 1: $p < N$. Hölder's inequality implies

$$\int_\Omega |u(x,t)|^\mu\,dx \leq \Big(\int_\Omega |u(x,t)|^2\,dx\Big)^{\frac{p}{N}}\Big(\int_\Omega \Big||u|^{1+\frac{m-1}{p}}\Big|^{\frac{Np}{N-p}}\,dx\Big)^{\frac{N-p}{N}}. \tag{67}$$

Set

$$\gamma = \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |u(x,t)|^2\,dx \qquad \text{and} \tag{68}$$

$$\Big(|u|^{1+\frac{m-1}{p}}\Big)_\Omega = \frac{1}{\operatorname{meas}\Omega}\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx. \tag{69}$$

Then

$$\begin{aligned}\int_\Omega |u(x,t)|^\mu\,dx \leq C\gamma^{\frac{p}{N}}\Big(\int_\Omega \Big||u|^{1+\frac{m-1}{p}} - \Big(|u|^{1+\frac{m-1}{p}}\Big)_\Omega\Big|^{\frac{Np}{N-p}}\,dx\Big)^{\frac{N-p}{N}} \\ + C\gamma^{\frac{p}{N}}\Big(\int_\Omega \Big|\Big(|u|^{1+\frac{m-1}{p}}\Big)_\Omega\Big|^{\frac{Np}{N-p}}\,dx\Big)^{\frac{N-p}{N}}.\end{aligned} \tag{70}$$

We can apply the Sobolev embedding theorem to the first term of (70) to find

$$\int_\Omega |u(x,t)|^\mu\,dx \leq C\int_\Omega \Big|\nabla|u(x,t)|^{1+\frac{m-1}{p}}\Big|^p\,dx + C\Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx\Big)^p \tag{71}$$

where C depends only on N, p, m, Ω, and γ. What follows next depends upon the value of $1 + \frac{m-1}{p}$.

Case 1A: $p < N$ and $1 + \frac{m-1}{p} \leq 2$. In this instance, we can apply Hölder's inequality to the last term of (71), and use the fact that Ω is bounded, to obtain our result.

Case 1B: $p < N$ and $1 + \frac{m-1}{p} > 2$. In this case, we shall use the same interpolation inequality that we used in (62). Making this substitution, we find that

$$\int_\Omega |u(x,t)|^\mu \, dx \leq C \int_\Omega \Big| \nabla |u|^{1+\frac{m-1}{p}} \Big|^p \, dx + C \Big(\int_\Omega |u(x,t)|^\mu \, dx \Big)^D \tag{72}$$

where C depends on γ and

$$D = \big(\frac{1-\omega}{\mu}\big)\big(1 + \frac{m-1}{p}\big)p = (1-\omega)\frac{p + (m-1)}{\frac{N+2}{N}p + (m-1)} \tag{73}$$

so that $0 < D < 1$. We can then apply Young's inequality to the last term of (72); our claim will follow upon an integration in time.

Case 2: $p \geq N$. Now

$$\begin{aligned} &\int_\Omega |u(x,t)|^\mu \, dx = \int_\Omega \Big| |u(x,t)|^{1+\frac{m-1}{p}} \Big|^q \, dx \\ &\leq C \int_\Omega \Big| |u(x,t)|^{1+\frac{m-1}{p}} - \Big(|u|^{1+\frac{m-1}{p}} \Big)_\Omega \Big|^q \, dx + C \int_\Omega \Big| \Big(|u|^{1+\frac{m-1}{p}} \Big)_\Omega \Big|^q \, dx \end{aligned}$$

where q is still defined by (52). Once more we can apply the full version of the Sobolev embedding theorem [4, Chapter 1, Theorem 2.1] to see that

$$\begin{aligned} &\int_\Omega |u(x,t)|^\mu \, dx \leq C \Big(\int_\Omega \Big| |u(x,t)|^{1+\frac{m-1}{p}} - \Big(|u|^{1+\frac{m-1}{p}} \Big)_\Omega \Big|^s \, dx \Big)^{\frac{q}{s}(1-\theta)} \\ &\times \Big(\int_\Omega \Big| \nabla \Big[|u|^{1+\frac{m-1}{p}} - \Big(|u|^{1+\frac{m-1}{p}} \Big)_\Omega \Big] \Big|^p \, dx \Big)^{\frac{q\theta}{p}} + C \int_\Omega \Big| \Big(|u|^{1+\frac{m-1}{p}} \Big)_\Omega \Big|^q \, dx \end{aligned} \tag{74}$$

where θ satisfies (55) and s is chosen so that (54) and (56) still hold. What happens next depends upon the value of $1 + \frac{m-1}{p}$.

Case 2A: $p \geq N$ and $1 + \frac{m-1}{p} \leq 2$. We again set $s = \frac{2}{1+\frac{m-1}{p}}$; we have already seen that this choice of s makes (54) and (56) true. Making this

substitution, we see that (74) becomes

$$\int_\Omega |u(x,t)|^\mu\,dx \le C\Big\{\int_\Omega |u(x,t)|^2\,dx + \Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx\Big)^s\Big\}^{\frac{p}{N}} \times \Big(\int_\Omega \Big|\nabla |u|^{1+\frac{m-1}{p}}\Big|^p\,dx\Big) + C\Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx\Big)^q \tag{75}$$

where we have used (57). Since $1+\frac{m-1}{p} \le 2$, we can use Hölder's inequality and then integrate the result in time to obtain our result.

Case 2B: $p \ge N$ and $1+\frac{m-1}{p} > 2$. In this instance, we again set $s=1$. As before, we have shown in (59) and (60) that the conditions (54) and (56) are satisfied. Then with this choice of s, (74) becomes

$$\int_\Omega |u(x,t)|^\mu\,dx \le \Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx\Big)^{q(1-\theta)}\Big(\int_\Omega \Big|\nabla |u|^{1+\frac{m-1}{p}}\Big|^p\,dx\Big)^{\frac{q\theta}{p}} + C\Big(\int_\Omega |u(x,t)|^{1+\frac{m-1}{p}}\,dx\Big)^q. \tag{76}$$

We again apply the interpolation inequality (62) with $0<\omega<1$ satisfying (63) to see that

$$\begin{aligned}\int_\Omega |u(x,t)|^\mu\,dx \le C\Big\{&\Big(\int_\Omega |u(x,t)|^2\,dx\Big)^{\frac{\omega}{2}(1+\frac{m-1}{p})} \\ &\times \Big(\int_\Omega |u(x,t)|^\mu\,dx\Big)^{\frac{1-\omega}{\mu}(1+\frac{m-1}{p})}\Big\}^{q(1-\theta)}\Big(\int_\Omega \Big|\nabla |u|^{1+\frac{m-1}{p}}\Big|^p\,dx\Big)^{\frac{q\theta}{p}} \\ &+ C\Big\{\Big(\int_\Omega |u(x,t)|^2\,dx\Big)^{\frac{\omega}{2}(1+\frac{m-1}{p})}\Big(\int_\Omega |u(x,t)|^\mu\,dx\Big)^{\frac{1-\omega}{\mu}(1+\frac{m-1}{p})}\Big\}^q.\end{aligned} \tag{77}$$

Letting C depend upon γ, we see that

$$\begin{aligned}\int_\Omega |u(x,t)|^\mu\,dx\,dt \le C\Big\{&\Big(\int_\Omega |u(x,t)|^\mu\,dx\Big)^{\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q(1-\theta)} \\ &\times \Big(\int_\Omega \Big|\nabla |u|^{1+\frac{m-1}{p}}\Big|^p\,dx\Big)^{\frac{q\theta}{p}}\Big\} + C\Big(\int_\Omega |u(x,t)|^\mu\,dx\Big)^{\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q}.\end{aligned} \tag{78}$$

Dividing, we obtain the estimate

$$\int_\Omega |u(x,t)|^\mu dx \le C\Big(\int_\Omega \big|\nabla |u(x,t)|^{1+\frac{m-1}{p}}\big|^p dx\Big)^B + C\Big(\int_\Omega |u(x,t)|^\mu dx\Big)^D \tag{79}$$

where $B = \frac{q\theta}{p}\frac{1}{1-\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q(1-\theta)} = 1$ and $D = \frac{\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q\theta}{1-\frac{1-\omega}{\mu}(1+\frac{m-1}{p})q(1-\theta)}$ $= (1-\omega)\frac{p+(m-1)}{\frac{N+2}{N}p+(m-1)}$ is the same number we found in (73). Because $0 < D < 1$, we can apply Young's inequality to the last term of (79) and then integrate in time to obtain our result.

Proof of Proposition 3. Our last task is to prove Proposition 3. Set $w(x,t) \equiv (u-k)_+ + k$ for some $k \geq C_7$. Note that

$$w(x,t) = \begin{cases} u(x,t) & \text{if } u(x,t) > k, \\ k & \text{if } u(x,t) \leq k, \end{cases} \tag{80}$$

and consequently that $\nabla w = 0$ in the set $[u \leq k]$. Now if $\Omega' \times (t_1, t_2) \Subset \Omega_T$, then

$$\begin{aligned} \operatorname*{ess\,sup}_{t_1<t<t_2} \int_{\Omega'} |w(x,t)|^2 \, dx &= \operatorname*{ess\,sup}_{t_1<t<t_2} \int_{\Omega'} \Big[(u(x,t)-k)_+ + k\Big]^2 \, dx \\ &\leq k^2 \operatorname{meas} \Omega' + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_{\Omega'} |u(x,t)|^2 \, dx < \infty. \end{aligned} \tag{81}$$

Further

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\Omega'} |w|^{m-1} |\nabla w|^p \, dx \, dt &= \int_{t_1}^{t_2} \int_{\Omega'} |u|^{m-1} |\nabla u|^p \chi[u > k] \, dx \, dt \\ &\leq \frac{1}{C_6} \int_{t_1}^{t_2} \int_{\Omega'} \Phi(|u|) |\nabla u|^p \, dx \, dt < \infty. \end{aligned} \tag{82}$$

Together, these allow us to apply Proposition 5 to w to conclude that

$$\iint_{[u>k] \cap \left(\Omega' \times (t_1,t_2)\right)} |u(x,t)|^\mu \, dx \leq \int_{t_1}^{t_2} \int_{\Omega'} |w(x,t)|^\mu \, dx \, dt < \infty. \tag{83}$$

A similar argument with $(u+k)_- - k$ replacing $(u-k)_+ - k$ shows us that

$$\iint_{[u<-k] \cap \left(\Omega' \times (t_1,t_2)\right)} |u(x,t)|^\mu \, dx < \infty \tag{84}$$

and the proposition follows.

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